

### Change of Perspective: Geometric Interpretation of a Leontief Economy

In the following paragraphs the point of view is moved away from a side by side representation of indicators of the branches of production of an economy towards a perspective where the indicators of all branches of production are represented in one single mathematical entity. It is possible to interpret e.g. a price system or gross output not only as a collection of numbers in a row, but at the same time as coordinates of a point in an  $n$ -dimensional space.  $n$  is the number of branches of production. This means that gross output can be represented as one point of an  $n$ -dimensional vector space<sup>1</sup> or more precisely as an  $n$ -dimensional vector. The space should have a *norm* to describe the length of each vector. We know that relative prices are defined only up to a constant factor. For this reason we can choose freely the length of the price vector, either as 1 or as the total turnover of all the economy (which would mean that  $x$  becomes the unit vector 1).

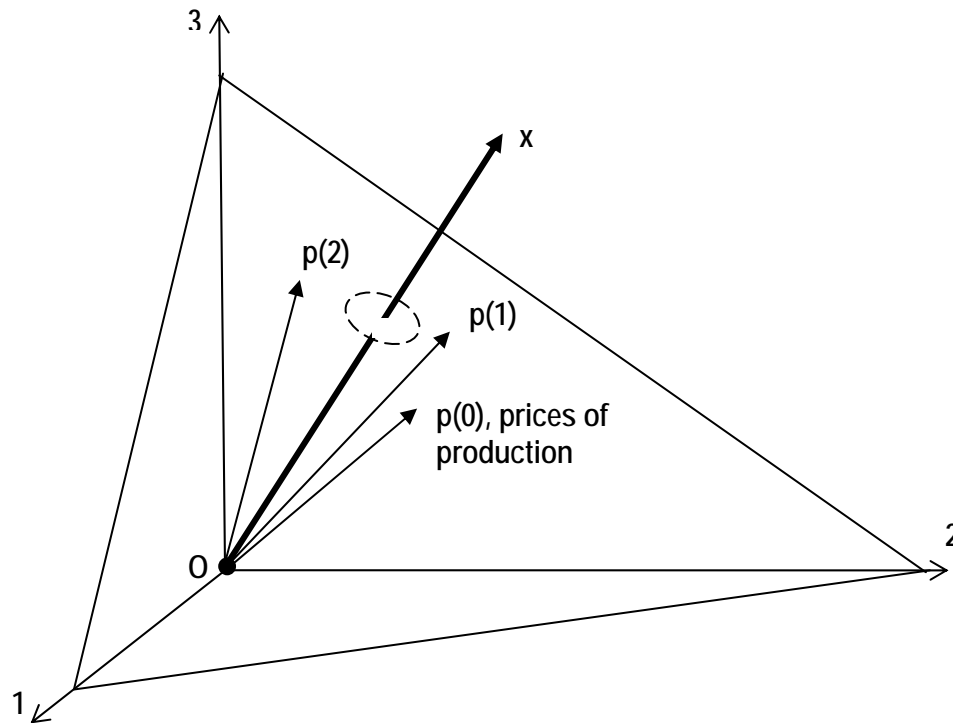


Figure 10: The location of all systems of relative prices  $p(i)$  on a hyperplane (three-dimensional case)

The condition implemented by Marx to transform values into prices<sup>2</sup> says that the sum of all the values before and after any transformation of price systems remains invariant ( $px = \text{const}$ ). For expressing this property we have to extend the concept of normed vector space by allowing for an inner product (this is nothing else than a generalization of the dot product of two vectors in analytical geometry. Two vectors are defined orthogonal if their inner product is zero). A normed vector space together with an inner product is called a Hilbert space. The geometric interpretation is easy: Marx' invariance condition means that the price vectors related to a given  $x$  are located

<sup>1</sup> Luenberger, D. G: Optimization by Vector Space Methods, Wiley, New York 1969, pp.11-45

<sup>2</sup> as assumed in the numerical examples of Chapter 9 of Marx' Capital's Volume III.

on a hyperplane (with a degree of freedom of  $n-1$ , where  $n$  is the number of branches of production). The hyperplane is orthogonal to the vector  $x$  (figure 10). We will come back to this hyperplane later.

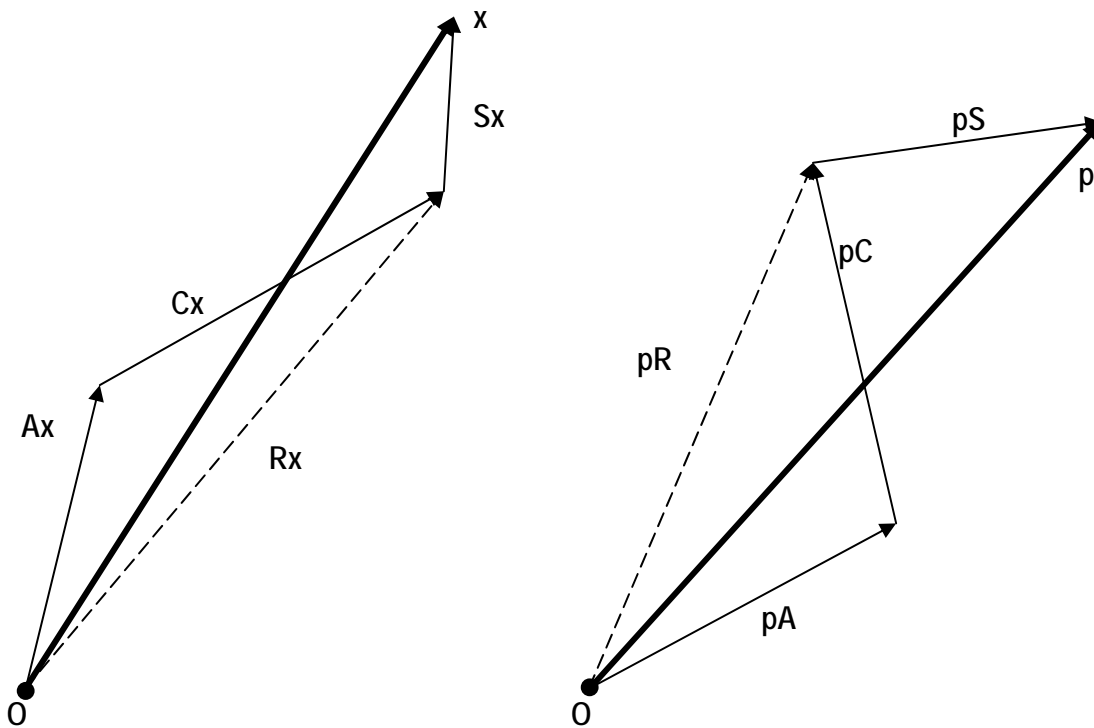


Figure 11: Decomposition of output  $x$  (left) and unit prices (right)

A second example of geometric interpretation is the decomposition of unit prices  $p$  and of the output  $x$ . If we construct a consumption matrix  $C$  (in a close analogy to the matrix  $A$  of technical coefficients containing the amounts of consumer goods of type  $i$  per unit of output  $j$ ), a “reproduction matrix”  $R = A + C$ , and a surplus matrix  $S$  (containing the amounts of investment goods of type  $i$  per unit of output  $j$  financed out of profits of sector  $j$ ), the equation

$$Ax + Cx + Sx = Rx + Sx = x$$

allows for a decomposition of  $x$  as a vector sum of intermediary goods  $Ax$ , consumption  $Cx$  and surplus product  $s = Sx$ . The dual equation

$$p = pA + pC + pS = pR + pS$$

allows for a decomposition of  $p$  as a vector sum of the various kinds of costs, constant capital  $pA$ , variable capital  $pC$  and surplus value  $pS$  (both dual geometric representations are shown in figure 11).

For prices of production which are the left-eigenvectors of the reproduction matrix  $R = A + C$  holds that  $p$ ,  $pR$  (the total unit costs of reproduction) and  $pS$  (the vector of unit surplus values or unit profits) are parallel in the  $n$ -dimensional vector space:

$$p = pR (1 + r) = pR + r pR = pR + pS$$

This means that the quotient of individual profit and capital advanced is equal for each branch of production.  $(1 + r)$  is nothing else than the inverse of the largest eigenvalue of the of reproduction matrix  $R$ .

According to the principle of duality the analogue to the prices of production can be presented also for the use-values: If  $x$  is the right eigenvector of the matrix of reproduction  $R$ ,  $x$  describes relative amounts of use-values necessary for equilibrium growth of the rate  $g$ . The largest eigenvalue of matrix  $R$  is unique, thus the rate of growth is equal to the rate of profit ( $r = g$ )

$$x = Rx (1 + g) = Rx + r Rx = Rx + Sx$$

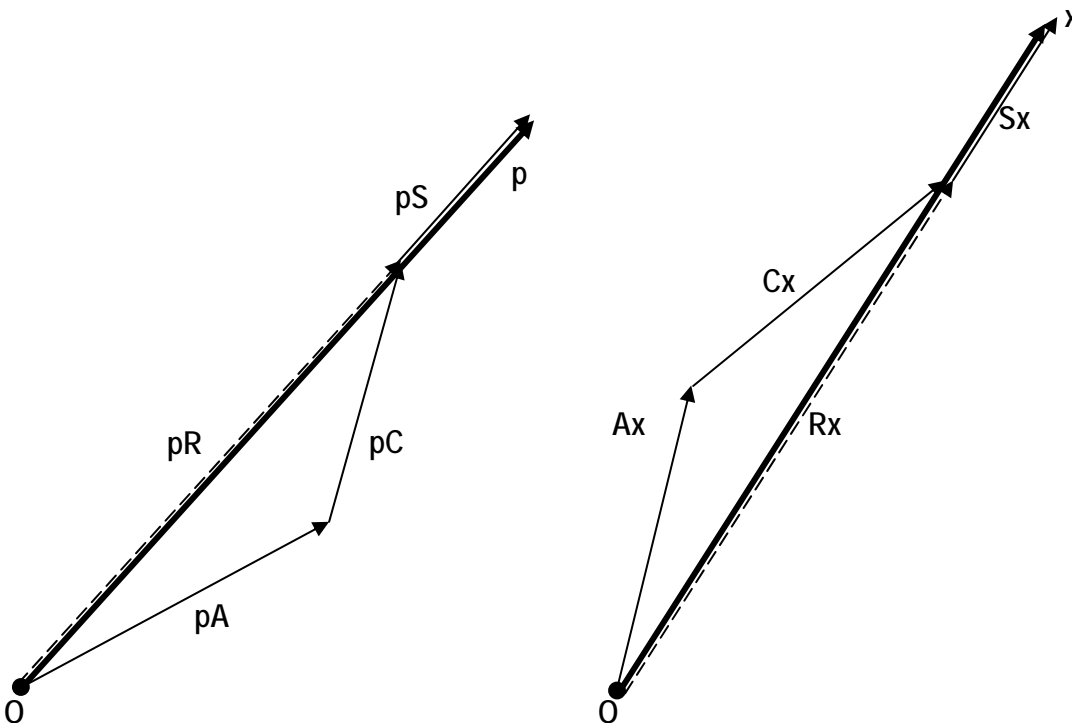


Figure 12: Decomposition of output  $x$  (left) for equilibrium growth and unit prices (right) for unit prices of production

These considerations can be used as a basis for the analysis of Marx' schemes of simple and extended reproduction (figure 12). After a short investigation of these schemes it becomes evident that Marx' assumptions were far from eigenvectors, and therefore the rates of growth he determined in his examples are far below the growth rates possible (according to his choice of the reproduction matrix  $R$ ). This is the reason why he had great difficulties to establish a clear cut analysis of equilibrium growth. But nevertheless he asked the appropriate questions.

If we choose the  $n$ -dimensional space as a space of turnover (in the dimension unit price times number of use-values), both aspects of duality can be presented in one single figure (figure 13). We draw Marx' variables „ $c$ “, „ $v$ “ and „ $m$ “ as vectors in an  $n$ -dimensional space and add them up as vectors to the total value „ $w$ “.

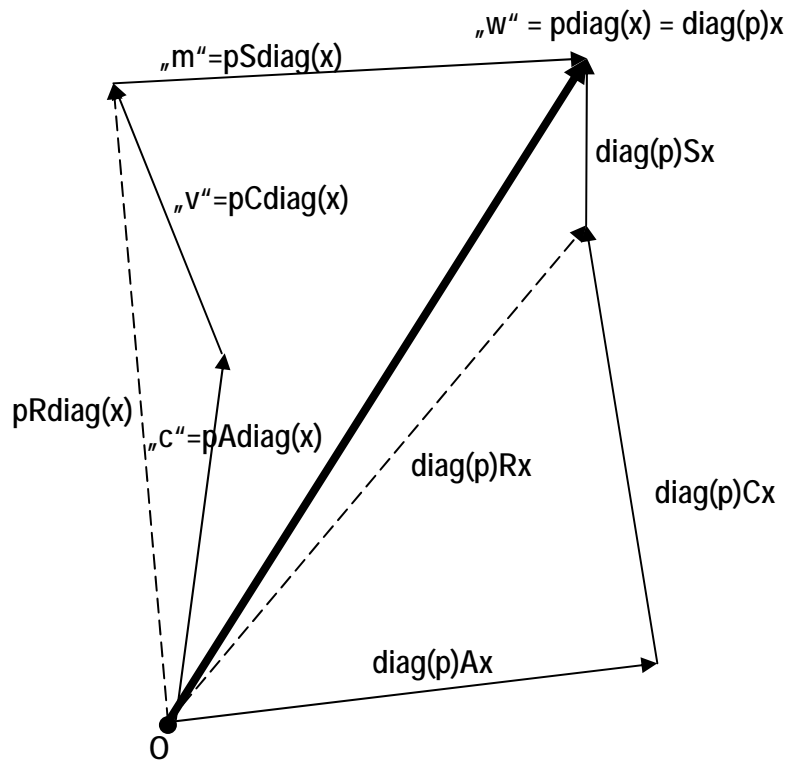


Figure 13: Decomposition of turnover (output  $x$  times unit price  $p$ )

The sums of the components of the dual vectors are always equal to each other, irrespective if they were derived from a perspective of use-values or unit prices:

$$1' \text{diag}(p)Ax = pA\text{diag}(x)1 = pAx$$

$$1' \text{diag}(p)Cx = pC\text{diag}(x)1 = pCx$$

$$1' \text{diag}(p)Sx = pS\text{diag}(x)1 = pSx$$

$$1' \text{diag}(p)x = p\text{diag}(x)1 = px$$

Because according to our assumptions matrices  $A$ ,  $C$  (and  $R = A + C$ ) and the vector  $x$  are given, only  $S$ , and – related to the choice of  $S$ ,  $p$  can be chosen. But even this choice cannot be done without restrictions: For each matrix  $S$  and unit price vector  $p$  (characterized by an index  $i$ ,  $i = 1, 2, \dots$ ) it holds that the total sum of profits  $p(i)S(i)\text{diag}(x)1$  has to be equal to the value  $p(i)s = 1' \text{diag}(p(i))S(i)x$  of the surplus product, where the surplus product  $s$

$$s = (E - R)x$$

is a given invariant.<sup>3</sup>

For each  $S(i)$

$$S(i)x = s,$$

i.e. the surplus product is invariant. And for each  $S(i)$  and unit price system  $p(i)$

$$p(i)S(i)x = p(i)s = p(i)(E - R)x,$$

i.e. the total value of the sum of profits has always to be equal to the total value of the surplus product. If profits or surplus values are known, we can construct a specific matrix of the surplus product  $S(i)$ . This is the case, whenever we start from known unit prices  $p(i)$ . We use the simplifying idea that the surplus product is proportionally allocated to the branches of production according to the specific surplus values in each of the branches of production. The surplus product itself remains constant.

$$\begin{aligned} S(i) &= s \cdot m(i) / m(i) \cdot 1 = s \cdot p(i)(E - R) / p(i)(E - R)x \\ &= (E - R) \cdot 1 \cdot p(i)(E - R) / p(i)(E - R)x \end{aligned}$$

Adversely, the unit prices  $p(i)$  are the left eigenvector of the matrix equation with eigenvalue 1

$$p(i) [R + S(i)] = p(i).$$

### **The interplay of price systems in geometric perspective**

A final remark is appropriate which is not really surprising within the context of matrix calculus, but in the context of the transformation problem it seems to be interesting: As already Samuelson has correctly observed, the rate of profit and the prices of production are determined simultaneously. If we go for an iterative method for solving the above equation, it is irrelevant from which price vector we start. The only pre-condition is that the start vector is a feasible price vector, meaning that it is located on the hyperplane shown in figures 10 and 14 and is also a solution<sup>4</sup> of the equation

$$p(i) [A + C + S(i)] = p(i).$$

We could start at the actual price system, or at labour values determined on the basis of all sectors or only on the sectors of material production, or at any other feasible price system: All iterations that apply Marx' method of allocating equal profit rates to each of the sectors (by linking total profits per unit capital advanced as a mark-up to the amount of capital advanced in each of the sectors) after a few iterations end up at the same vector of prices of production (see figure 14 for a two-dimensional illustration of the hyperplane. Dotted lines represent possible pathways of convergence toward prices of production within the hyperplane. For defining the precise shapes of these pathways further investigation is needed).

Therefore, in 1971 Paul Samuelson could write with some truth in it:

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<sup>3</sup> This invariance holds only for a given price system, it does not hold during the transformation of labour values into prices of production.

<sup>4</sup> This will create a subset of feasible price systems in the hyperplane. Its precise shape is not investigated here.

„Contemplate the two mutually-exclusive alternatives of 'values' and 'prices'. Write down one. Now transform by taking an eraser and rubbing it out. Then fill in the other one. *Voila!* You have completed your transformation algorithm.”<sup>5</sup>

In history and in other evolutionary processes it happens rather frequently that a certain traditional, older regime is replaced by a new one. In such a case the new system does not carry around any longer the mechanisms of its predecessor. It needs scientific methods to show the traces of the perished system within the surviving one. In the case of the transformation problem we can reconstruct the former value structure by mathematical transformations of the price system. Matrix algebra provides tools similar to a time machine to look back to the logical or maybe temporal history of the capitalistic structure of the surface.

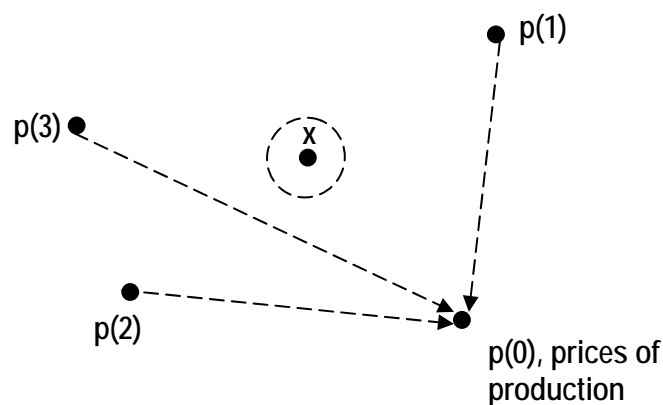


Figure 14: Two-dimensional illustration of the hyperplane and the location of all feasible price systems on it.

Although from a mathematical perspective it might be sufficient to observe actual prices only or to determine directly the numerical values of prices of production based on the reproduction matrix  $R$ . But this restricted view of the economy would separate any social scientist from a more comprehensive analysis, and it would leave the economist in an ontological vacuum. The quality (in a philosophic perspective) of prices cannot be linked any longer to essential human activities like work, and further considerations about the nature of the economy would be suppressed. Neither would it be possible to link economics to the non quantitative realms of anthropology, history or philosophy.

As Hegel and others have stated frequently, to understand any phenomenon in a deeper way you should know its history and origin. If we – like Samuelson – destroy all the links to the origin of prices, we destroy all the possibilities of deeper investigation.

If we keep up the road of logical or historical emergence of prices, we have to choose which price system we select as relevant for further investigation. Within the labour theory of value two variants of price systems can be offered: (1) values created only by means of material branches of production or (2) values created by all sectors of production of the economy. We can show that in

<sup>5</sup> Paul Samuelson: Understanding the Marxian Notion of Exploitation: A Summary of the So-Called Transformation Problem Between Marxian Values and Competitive Prices. *Journal of Economic Literature* 1971 9 2: 399–431

the second case the principle of equivalent exchange is violated, while in case (1) it is not. Thus, my preference would be to go back the pathway of abstraction from the phenomena at the surface in a consistent way and look for a theoretical description of value creation free of contradictions. This can only be done on the basis of the principle of equivalent exchange, meaning that the producers of labour value can acquire the same amount of value in terms of other use values in exchange.